**MA3025 Solutions Exam # 3**

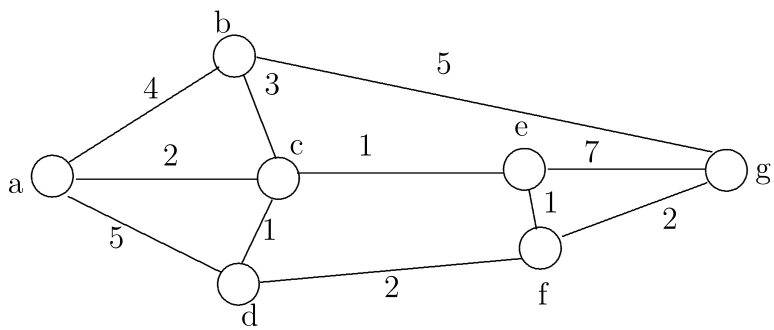
**Due 11am December** 10**th,** 2007 **Name** ________________________________

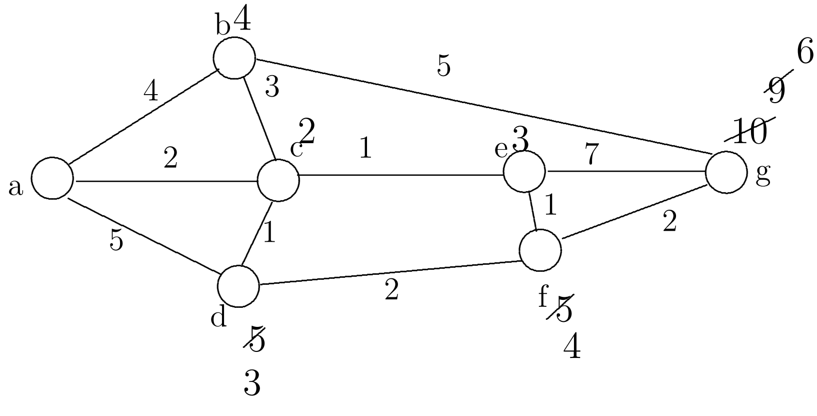
**Instructor: Dr. Ralucca Gera**

**Show all necessary work in each problem to receive credit. You may ONLY use your notes and Rosen book (no collaboration is allowed either).**

**1.** (10 points) Suppose that the vertex set of a graph $G$ is the set $V(G) = \{x_1, x_2, x_3, \ldots, x_{10,000}\}$. Two vertices $x_i$ and $x_j$ are adjacent if $i + j$ is odd. What is the graph $G$? (identify the class and/or name of the graph).

**2.** (10 points) Find the shortest path between $a$ and $g$ in the weighted graph below, using Dijkstra's Algorithm. Show your work (either a table or on the graph)

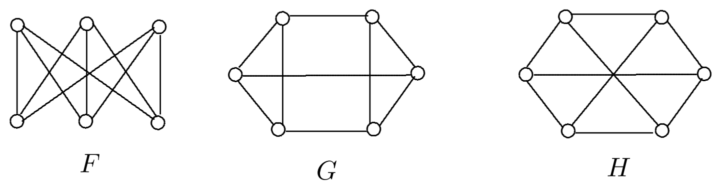



a shortest path: $a, c, d, f, g$ of length 6.

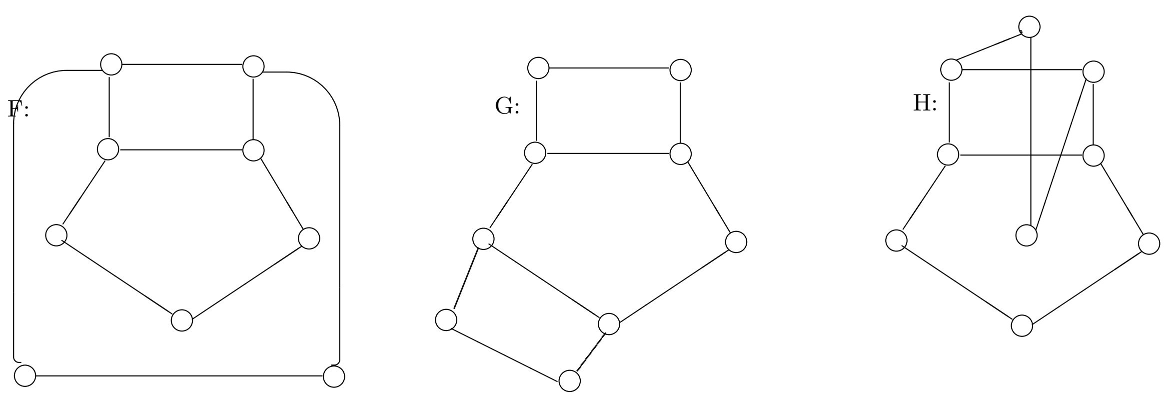
**3.** (20 points)

(a) Consider the three graphs $H$, $G$, $F$ below. Find two graphs in $\{H, G, F\}$ that are isomorphic, and two that are NOT isomorphic [explanations required].



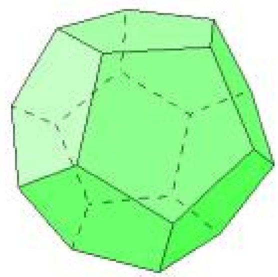

$F \cong H \cong K_{3,3}$
$G \ncong F$ since $G$ has a triangle and $F$ doesn't.

(b) Consider the three graphs $H$, $G$, $F$ below. Find two graphs in $\{F, G, H\}$ that are isomorphic, and two that are NOT isomorphic [explanations required].

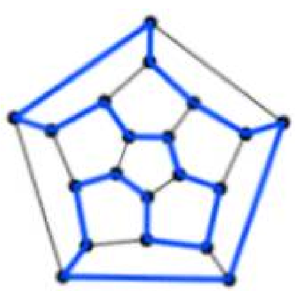



$F \cong H$
$G \ncong F$ since either:
(1) $G$ has the vertices of degree 3 on the 5-cycle and $F$ doesn't, or
(2) $F$ has three vertices of degree two that form a path on three vertices, and $G$ doesn't, or
(3) $F$ has a cycle of length 6 and $H$ doesn't.

**4.** (10 points) A dodecahedron is an 3 dimensional object that is equilateral (all sides are equal) and equiangular (all angles are equal) with 12 regular faces – this making it ideal as a desk-calendar paperweight– so that each face is a regular pentagon (see below). It is said that the famous Irish mathematician Sir William Rowan Hamilton invented a game which involved the dodecahedron. Hamilton labeled each vertex of the dodecahedron with the name of a well-known city. The object of the game was for the player to travel "around the world" by determining a round trip which included all the cities exactly once, with the added restriction that it is possible to travel from one face of the dodecahedron to the other if the two faces share a common boundary. Model this by a graph, and use a graph theory to solve the puzzle. Here is the dodecahedron:

Solution.



**5.** (50 points) Let $G_n$ be the complement of the cycle $C_n$ ($n \geq 3$).

a. (5 points) What is the degree sequence of $G_n$ ($n \geq 3$)?
Solution Since each vertex in $C_n$ has degree 2, it follows that every vertex in $G_n$ has degree $(n-1)-2$. Thus the degree sequence is $n-3, n-3, \ldots, n-3$.

b. (10 points) What is the chromatic number of $G_8$?
Solution Since $G$ contains a complete graph on 4 vertices (see the vertices colored $1, 2, 3$ and 4 below), it follows that $\chi(G_8) \geq 4$. Also, a coloring with 4 colors is presented below, showing that $\chi(G_8) \leq 4$. Therefore $\chi(G_8) = 4$.

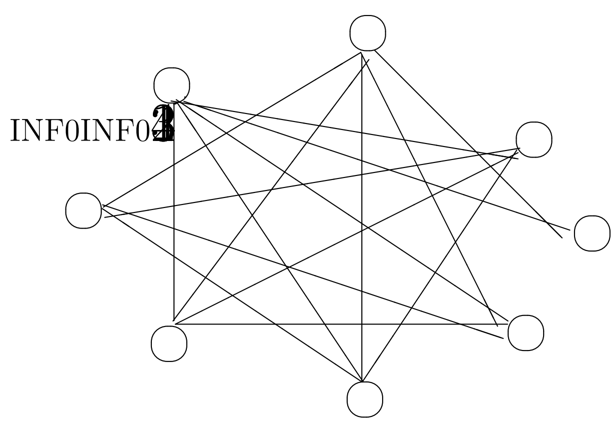



Figure 1: a proper coloring of $G_8$

c. (10 points) Determine with justifications, which graphs $G_n$ $(n \geq 3)$ have a Hamiltonian circuit and which ones do not.

Solution Since $G_3$ and $G_4$ are not even connected, they will not have a Hamiltonian circuit. $G_5 \cong C_5$, and so it will have a Hamiltonian circuit, namely the cycle itself. For $G_n, n \geq 6$, note that $\deg v = n - 3 \geq \lceil \frac{n}{2} \rceil$, and so it will have a Hamiltonian circuit. And so $G_n$ has a Hamiltonian circuit for $n \geq 5$.

d. (10 points) Determine with justifications, which graphs $G_n$ $(n \geq 3)$ have an Eulerian circuit and which ones do not.

Solution Since $G_3$ and $G_4$ are not even connected, they will not have an Eulerian circuit. For $G_n, n \geq 5$, note that $\deg v = n - 3$, which is even if $n$ is odd and $G$ connected. And so $G_n$ has a Hamiltonian circuit for odd values of $n \geq 5$.

e. (15 points) Determine with justifications, which graphs $G_n$ ($n \geq 3$) are planar and which ones are not.

<u>Solution</u> We claim that $G$ is planar for $3 \leq n \leq 6$, and it is not otherwise.
<u>Proof:</u> Note that the complement of $C_n$, $\bar{C}_n$, is a $n-3$-regular graph. Thus $e_{\bar{C_n}} = \frac{n(n-3)}{2}$.

If $n \geq 8$, then $e_{\bar{C_n}} = \frac{n(n-3)}{2} > 3n - 6$, thus being not planar. For $3 \leq n \leq 7$ see below:

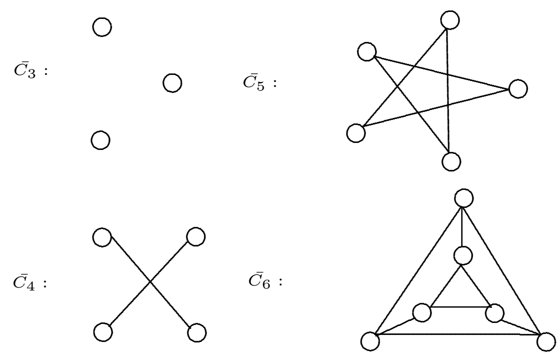



Figure 2: $\overline{C_n}$ is planar for $3 \leq n \leq 6$

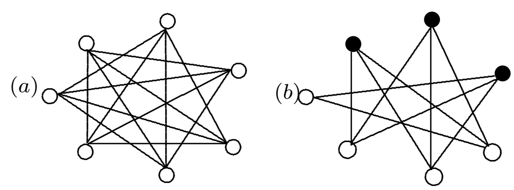


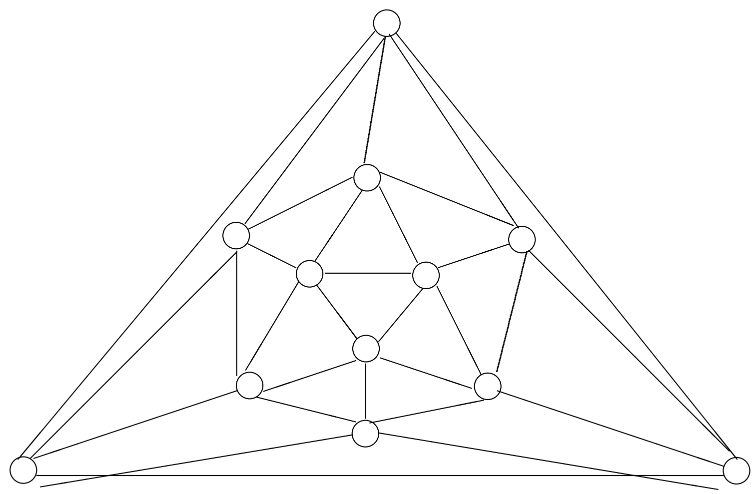

Figure 3: (a) $\bar{C}_7$, and (b) vertices of degree 3 form a $K_{3,3}$ subdivision subgraph of $\bar{C}_7$

Therefore, $G_n$ contains a copy of $K_{3,3}$ making it nonplanar. And so $G_n$ is only planar for $3 \leq n \leq 6$.

**6.** (EXTRA CREDIT: 5 points, choose one of the following two for 5 points)

- Find a polynomial time algorithm for the Traveling Salesman Problem.
- Find (and prove) the chromatic number for the only maximum planar regular graph on 12 vertices.

  Solution for (b):



The Chromatic number is at most 4 since it is planar (by the 4 color Thm.), and at least 4 colors are needed since it contains a wheel $W_{1,5}$.